Suspension of a measuring function *or* Sharing some vague thoughts on comparaison of methods

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Motivation

1. **General motivation:** We need good examples for testing and comparing new algorithms and programs in high dimensions.
   - Real life and digital data (e.g. from medical imaging): We do not yet understand the output, so not suitable for detecting some types of errors;
   - Constructions made by hand: either topology or numerics not enough challenging.
2. **Specific goal:** Getting insight into a current discussion on what is a better tool for shape comparison:
   - the *rank invariant* of *measuring functions* studied for all homology dimensions

     or
   - the *multiparameter size functions* (commonly called *multidimensional size functions*), where only numbers of connected components are involved.

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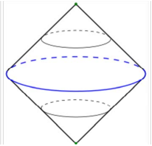
Suspension



$SX := X \times [-1,1]/\sim,\ (x,1) \sim (y,1), (x,-1) \sim (y,-1)$

S shifts the dimension of reduced homology by 1:

$$\tilde{H}_{q+1}(SX) \cong \tilde{H}_q(X),\ q \geq -1$$

Equivalence for non-reduced homology:

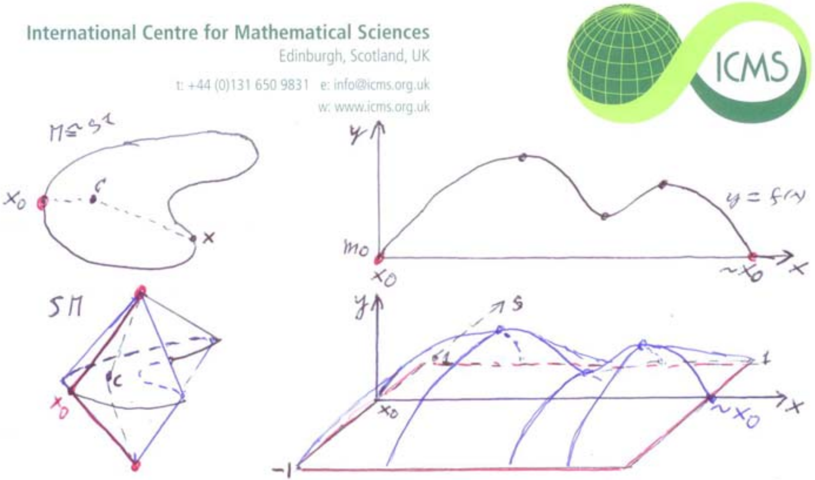
$$H_{q+1}(SX) \cong H_q(X),\ \text{for } q \geq 1,$$

$$H_1(SX) \oplus \mathbb{F} \cong H_0(X),\ \text{and } H_0(SX) \cong \mathbb{F}.$$

The *based suspension* of (X, x_0) is $(\Sigma X, (x_0, 0))$ with an additional identification $(x_0, s) \sim (x_0, 0)$ for all $s \in [-1, 1]$.

$$\Sigma X \cong SX$$

$\mathbb{R}$–target suspension of f



$X = M$ connected compact manifold, $f : M \to \mathbb{R}$ *Morse function*.
Define $S_{\mathbb{R}} f : SM \to \mathbb{R}$ by

$$S_{\mathbb{R}} f(x, s) := s^2 m_0 + (1 - s^2) f(x).$$

Proposition

(p, s) is a critical point of $S_{\mathbb{R}} f \iff$ either p is a critical point of f and $s = 0$, or $s = \pm 1$. In the first case, p is of Morse index λ $\iff$ $(p, 0)$ is of Morse index $\lambda + 1$.

Note: Passing to $\Sigma_{\mathbb{R}} f$ restores the isolation condition for $(x_0, 0)$.

Sublevel sets

$$M_\alpha := \{x \in M \mid f(x) \leq \alpha\} \ , \alpha \in \mathbb{R}, \text{ and}$$

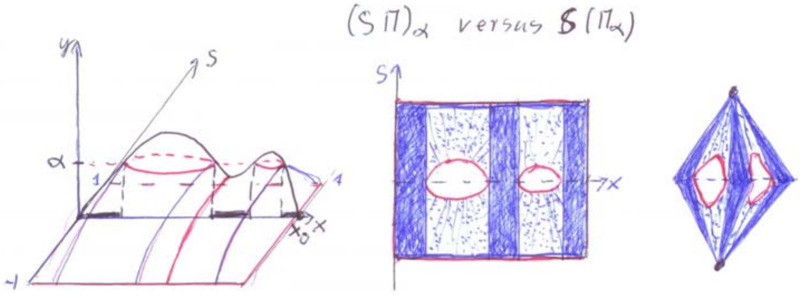
$$(SM)_\alpha := \{(x, t) \in SM \mid S_{\mathbb{R}} f(x) \leq \alpha\}$$

Note: $S(M_\alpha) \subset (SM)_\alpha$.

Lemma

We have the homotopy equivalence

$$S(M_\alpha) \cong (SM)_\alpha.$$



Given $\alpha \leq \beta$ in $\mathbb{R}$, the inclusion $j^{(\alpha,\beta)} : M_\alpha \hookrightarrow M_\beta$ induces

$$H_*(j^{(\alpha,\beta)}) : H_*(M_\alpha) \to H_*(M_\beta)$$

The q–th *rank invariant* of (M, f) is the function $\rho_f^q : \Delta_+ \to \mathbb{N}$,

$$\rho_f^q(\alpha, \beta) = \operatorname{rank} \operatorname{im} H_q(j^{(\alpha,\beta)}).$$

defined on

$$\Delta_+ := \{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha < \beta\}.$$

Set $\rho_f := \rho_f^*$. The *reduced rank invariant*:

$$\tilde{\rho}_f(\alpha, \beta) = \operatorname{rank} \operatorname{im} \tilde{H}_*(j^{(\alpha,\beta)}).$$

Theorem

For any $(\alpha, \beta) \in \Delta_+$,

$$\tilde{\rho}^{q+1}_{S_{\mathbb{R}} f}(\alpha, \beta) = \tilde{\rho}^{q}_{f}(\alpha, \beta), \; q \geq -1.$$

For the non-reduced rank invariant, we have

$$\rho^{q+1}_{S_{\mathbb{R}} f}(\alpha, \beta) = \rho^{q}_{f}(\alpha, \beta), \; q \geq 1,$$

$$\rho^{1}_{S_{\mathbb{R}} f}(\alpha, \beta) = \rho^{0}_{f}(\alpha, \beta) - 1,$$

and

$$\rho^{0}_{S_{\mathbb{R}} f}(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha < m_0, \\ 1 & \text{otherwise.} \end{cases}$$

The same holds for $\Sigma_{\mathbb{R}} f$.

Shape comparaison

$f : M \to \mathbb{R}$, $g : N \to \mathbb{R}$ *measuring functions*
$D(\rho_f, \rho_g)$ a stable distance, e.g. *matching distance* d_{match}

Theorem

$$\text{We have} \quad D(\tilde{\rho}_{S_{\mathbb{R}} f}, \tilde{\rho}_{S_{\mathbb{R}} g}) = D(\tilde{\rho}_f, \tilde{\rho}_g).$$

For non-reduced rank invariants:

$$D(\rho^{q+1}_{S_{\mathbb{R}} f}, \rho^{q+1}_{S_{\mathbb{R}} g}) = D(\rho^q_f, \rho^q_g), \quad q \geq 1, \quad \text{and}$$

$$D(\rho^1_{S_{\mathbb{R}} f}, \rho^1_{S_{\mathbb{R}} g}) \leq D(\rho^0_f, \rho^0_g).$$

The last inequality becomes equality, if f and g are normalized *i. e. their range is set to be $[0, 1]$.*

Note: No equality for $q = 0$, because *cornerlines* in the *presistence diagram* (equiv. infinite *barcode* intervals) at $\min f$ and $\min g$ disappear with S.

Multiparameter measuring functions

Almost all extends to $f : M \to R^k$ and $g : N \to R^k$. We get

- *$\mathbb{R}^k$–target suspensions*

  $S_{\mathbb{R}^k} f : M \to R^k$ and $S_{\mathbb{R}^k} g : N \to R^k$;

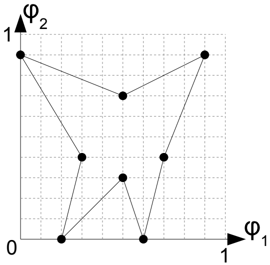
- $\tilde{\rho}^{q+1}_{S_{\mathbb{R}^k} f}(\alpha, \beta) = \tilde{\rho}^{q}_{f}(\alpha, \beta), \ q \geq -1$;

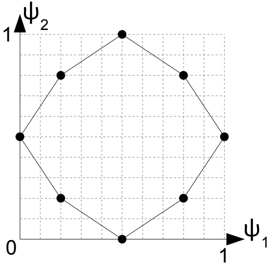
- $D(\tilde{\rho}_{S_{\mathbb{R}^k} f}, \tilde{\rho}_{S_{\mathbb{R}^k} g}) = D(\tilde{\rho}_f, \tilde{\rho}_g)$.

Note: Normalizing f_i and g_i to $[0, 1]$, $i = 1, 2, \ldots, k$, does not necessarily bring equality for non-reduced matching at $q = 0$.

Tests of 2D functions on curves

Letters ϕ, ψ used for piecewise-linear approximations of f, g.

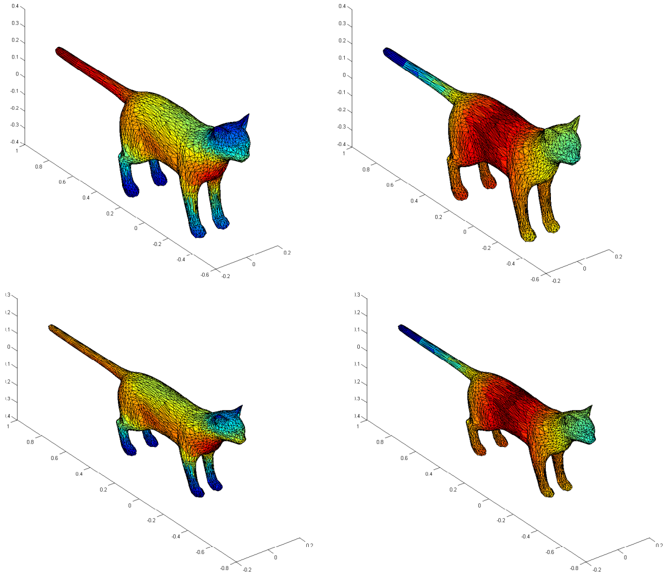




d_{match}	$i = 1$	$i = 2$	(φ, ψ)
$q = 0$ at M	0.05	0.15	0.20
$q = 1$ at SM	0.05	0.15	0.15

$$d_{match}(\tilde{\rho}^0_{\varphi}, \tilde{\rho}^0_{\psi}) = 0.15 = d_{match}(\rho^1_{S_{\mathbb{R}}\varphi}, \rho^1_{S_{\mathbb{R}}\psi})$$

Tests of 2D functions on surfaces



2D test results

d_{match}	$i = 1$	$i = 2$	(φ, ψ)
$q = 0$ at M	0.118165	0.032043	$0.225394 \pm \epsilon$
$q = 1$ at SM	0.118165	0.032043	0.144274
$q = 1$ at M	0.031129	0.039497	$0.225394 \pm \epsilon$
$q = 2$ at SM	0.031129	0.039497	$0.046150 \pm \epsilon$

where $\epsilon < 0.12$

$$d_{match}(\tilde{\rho}^0_\varphi, \tilde{\rho}^0_\psi) = 0.144274 = d_{match}(\rho^1_{S_{\mathbb{R}}\varphi}, \rho^1_{S_{\mathbb{R}}\psi})$$