A new type of a locally finite topological space and its applications

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July 6, 2012

Goals of this talk

Question: Is there a topological structure which can be used for studying both continuous and digital spaces?
As an answer, this talk suggests a new type of locally finite topological space (or a space set topology, *SST* for short).

Applications and properties:
(1) Classical, computer, discrete and digital geometry, computational topology as well as digital topology
(2) Jordan curve theorem of a simple closed *SST*-curve
(3) An *SST* satisfies the semi-$T_{\frac{1}{2}}$ separation axiom
(4) An *SST* has an asymmetric topological structure.

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Recent works related to this approach

1. S.E. Han, Continuity of maps between axiomatic locally finite spaces and its applications, *International Journal of Computer Mathematics* **88**(14) (2011) 2889-2900.
2. S.E. Han and V. Kovalevsky, A new type of locally finite topological space and its applications, submitted to Proceedings of the Royal Soc. of Math..
3. V. Kovalevsky, Finite topology as applied to image analysis, *Computer Vision, Graphics, and Image Processing* **46** (1989) 141-161.
4. V. Kovalevsky, Axiomatic Digital Topology, *Journal of Mathematical Imaging and Vision* **26** (2006) 41-58.
5. H. Seifert and W. Threlfall, A Textbook of Topology, Academic Press, 1980.

Neighborhood space

Let us consider a neighborhood space as a pair $S = (E, U)$ in the classical textbook by Seifert and Threlfall, where E is a nonempty set and U is a system of subsets of E, with the property that each element e of E is contained in some element of U, and that each such set belonging to U and containing e is called a neighborhood of e.
Based on the original version of an AC complex developed by Kovalevsky, we define below the notion of an abstract cell complex on the basis of a neighborhood relation.

AC complex

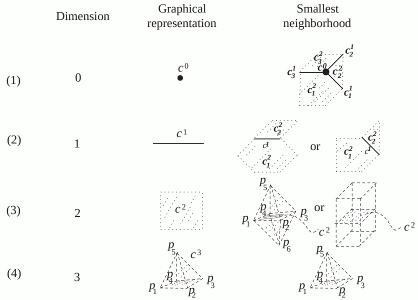
Definition

An abstract cell (for short, AC) complex $C = (E, N, dim)$ is a nonempty set E of abstract elements provided with
(1) a reflexive, antisymmetric and transitive binary relation $N \subset E \times E$ called the neighborhood relation,
(2) a dimension function dim: $E \to I$ from E into the set I of non-negative integers such that if $(a, b) \in N$ (i.e. a is an element of a neighborhood of b), then $dim(a) \geq dim(b)$.

Elements c_j^i of $E = \{c_j^i \mid i \in M, j \in M' \text{ and } |M'| \textit{ is finite}\}$ are called cells and the superscript i of the cell means its dimension, the subscript j of the cell means the only index for discriminating the i-dimensional cells, and the index sets M and M' depend on the situation.

(1) As for the terminology "abstract element" of an *AC* complex, note that a cell of an abstract cell complex, unlike a Euclidean cell or a simplex, is never a subset of another cell, which implies that an *AC* complex is different from a simplicial complex.
(2) In this state the neighborhood of an *AC* complex does not require a topological structure.

AC complex, Smallest neighborhood of an element of an *AC* complex(Fig.1)



Let us define the notion of smallest neighborhood of an element of an *AC* complex.

Definition

Let $C = (E, N, dim)$ be an *AC* complex. For an element $a \in E$ let $N(a, E) := N(a) = \{b \in E \,|\, (b, a) \in N\}$. Further, let $SN(a, E) := SN(a) = \cap N(a)$. Then we call $SN(a)$ a smallest neighborhood of a in E.

In this state we need to point out that the current smallest neighborhood is used without topology. Instead, it is only derived from an *AC* complex.

Let us now define the terminology "adjacent (or joins)" between two cells of an *AC* complex, as follows:

Definition

Let $C = (E, N, dim)$ be an *AC* complex. For two distinct elements a and b in E we say that a is adjacent to (or joins) b if $a \in SN(b)$ or $b \in SN(a)$.

Let us define the notion of smallest neighborhood of an element of an AC complex.

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Further information on an AC complex

Definition

Let $C = (X, N, dim)$ be an AC complex, where $X := \{c_j^i \,|\, i \in M, j \in M'\}$. For each proper m-cell c^m its boundary, denoted by $\partial(\{c^m\}) := \partial c^m$, is defined as follows: $\partial c^m := \{c_j^i \mid c_j^i$ is adjacent to (or joins) $c^m, i \lneq m\}$.

In this state the notion of the current boundary is not a topological boundary.

Remark

For a 0-cell (or a point) c^0 we say that $\partial c^0 = \emptyset$.

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Elementary subdivision developed by Kovalevsky

Definition

Let c^m be an m-cell of an nD AC complex, $1 \leq m \leq n$. An elementary subdivision of the m-cell c^m replaces the cell c^m by two proper m-cells c_1^m, c_2^m and one proper $(m-1)$-cell c^{m-1} whose smallest neighborhood $SN(c^{m-1})$ contains both m-cells c_1^m and c_2^m, while the cells c_1^m, c_2^m and c^{m-1} satisfy the conditions:
1) $\partial(\{c_1^m\} \cup \{c^{m-1}\} \cup \{c_2^m\}) = \partial c^m$ or it is a subdivision of ∂c^m, where ∂c^m means the boundary of the cell c^m;
2) $c^{m-1} \notin \partial c^m$;
3) $\partial c^{m-1} \subset \partial c^m$ or ∂c^{m-1} is a subset of the subdivision of ∂c^m.

Example

Figure 2(a) and (b) show a process of an elementary subdivision of a proper 2-cell. The emphasized points c_1^0 and c_2^0 in Figure 2(b) compose the 0-sphere lying in the boundary ∂c^2. The 0-sphere is spanned by the 1-cell c^1. The original cell c^2 (or an open 2-cell) is replaced in Figure 2(b) by the complex $\{c_1^2\} \cup \{c^1\} \cup \{c_2^2\}$ whose boundary is the same as that of c^2.

Motivation of a development of *SAC* complex

In order to proceed a special kind of subdivision of an *AC* complex, we need to recall tilings of $\mathbf{R}^n, n \in \mathbf{N}$, as follows: The real plane can be a *highly symmetric* tiling made up of *congruent regular* polygons. Only three kinds of regular tilings exist: those made up of equilateral triangles, squares or hexagons. An *edge-to-edge* tiling of a subplane in $\mathbf{R}^2$ is even less regular. The only requirement is that adjacent tilings only share *full* sides. Similarly, we can consider a *face to face quasicrystallization* of the 3D real space and further, their analogy to $\mathbf{R}^n, n \geq 4$. Motivated by these tilings and a (barycentric) subdivision of simplicial complexes, we can establish the following:

In Figure 2(a) we can observe that ∂c^2 consists of five 1-cells and five 0-ones surrounding the 2-cell c^2.
The notion of a subdivision of a given complex was often used in geometric topology, as in the case of a triangulation, by means of a series of elementary subdivisions of its cells. Kovalevsky established the notion of subdivision of an AC complex. More precisely, for an m-cell c^m of an nD AC complex with $1 \lneq m \leq n$ Kovalevsky proceeded to subdivide the m-cell c^m. Let us now suggest the correspondingly modified notion of elementary subdivision, as follows:

Subdivided AC complex, briefly, SAC complex

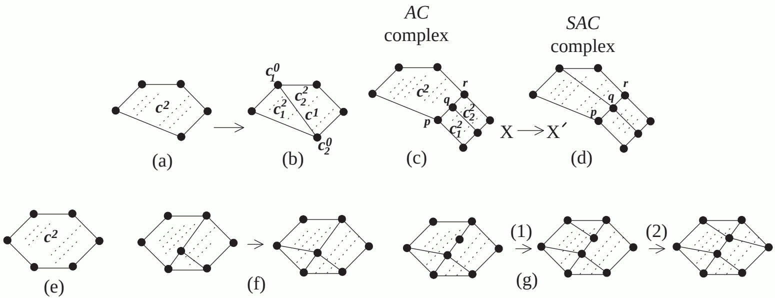
Definition

Let $C = (X, N, dim)$ be an AC complex, where $X := \{c_j^i \,|\, i \in M, j \in M'\}$. Proceed subdivisions of some m-cells in X, which is an AC complex denoted by $X' := \{c_{j_t}^i \,|\, i \in M_1, j_t \in M''\}$ with $M \subset M_1$ and $M' \subset M''$, such that there is no i-cell $c_{j_1}^i$ in X' satisfying that $\partial c_{j_1}^i$ is partially matched with $\partial c_{j_2}^i$, where $j_1 \neq j_2$ in M''. Then we call the subdivided AC complex an SAC complex.

Example

Consider the *AC* complex X of Figure 2(c). Then we obtain an *SAC* complex X' derived from X. More precisely, while the object X is an *AC* complex, it cannot be an *SAC* complex because some boundary of the 2-cells c_1^2 and c_2^2 are partially matched with ∂c^2 (see the 1-cells represented by the line segments pq and qr). Similarly, Figure 2(f) and (g) show special kinds of subdivisions of c^2 of Figure 2(e).

(a)-(b): Configuration of a subdivision of the 2-cell c^2; (c)-(d) and (f)-(g): Processes on constructing *SAC* complexes in terms of subdivisions.(Fig.2)



Based on the smallest neighborhood, for an *SAC* complex X we obtain a smallest neighborhood of an element $c_j^i \in X$ as follows:

Proposition

Let $C = (X, N, dim)$ *be an SAC complex, where* $X := \{c_j^i \,|i \in M, j \in M'\}$*. For each cell* $c_j^i \in X$ *we obtain its smallest neighborhood on* X *as follows:*

$$SN(c_j^i) = \{c_j^i, c_{j_1}^{i_1} \mid c_{j_1}^{i_1} \text{ is adjacent to (or joins) } c_j^i, i_1 \gneq i\}.$$

The neighborhood relation *SN* fulfils the notion of an *AC* complex such as it is reflexive, transitive and antisymmetric. Hereafter, we will use this smallest neighborhood suggested which will be used for establishing a topology on an *SAC* complex.

Example of SN of an element

In Figure 1, according to the dimensions 0, 1, 2 and 3, we observe the corresponding smallest neighborhoods of given cells. In Figure 1(1) and (2) we obtain $SN(c^0) = \{c_0, c_i^1, c_i^2 \mid i \in [1,3]_{\mathbf{Z}}\}$, $SN(c^1) = \{c_1, c_1^2, c_2^2\}$ (see Figure 1(2)) and so forth. In particular, assume that the octahedron without boundary in Figure 1(3) is formulated by the six points $p_i, i \in [1,6]_{\mathbf{Z}}$. Then it is exactly $SN(c^2)$, where c^2 is generated by the four points $p_i, i \in [1,4]_{\mathbf{Z}}$. In Figure 1(4) consider the proper tetrahedron c^3 generated by the five points $p_i, i \in [1,5]_{\mathbf{Z}}$. Then $SN(c^3)$ is the whole object itself of Figure 1(4).

By using the smallest neighborhood mentioned in Proposition 2.5, we now formulate a topology on an *SAC* complex X, named by a *space set topology* on X, as follows:

Definition

Let $C = (X, N, dim)$ be an *SAC* complex. Let $S := (X, U)$ be a binary set, where U is the set of all $SN(x), x \in X$. Then we obtain the topology on X induced by the set U as a base, denoted by (X, T). Further, we call this topology T a space set topology on X, briefly *SST* on X.

Remark

(1) If $|X| \geq 2$, then a connected SST (X, T) cannot be a discrete topological space.
(2) For an SST (X, T) consider two distinct elements x and y which are not adjacent to each other. Then there are smallest open neighborhoods of the elements, denoted by $SN(x)$ and $SN(y)$ in T, such that $y \notin SN(x)$ and $x \notin SN(y)$.

Jordan Curve Theorem of a Simple Closed SST-curve on the Real Plane with an Edge to Edge Tiling

Motivated by the classical Jordan curve theorem of the simple closed curve on the real plane, we can establish Jordan curve theorem of a simple closed SST-curve on the real plane with an edge to edge tiling, which has its own property different from the classical one because SST has its own topological structure.

For an *SST* we can consider the notions of *interior, exterior* and *boundary* as the corresponding notions from classical topology. By using the above properties, for an *SST*, (X, T), and a subset $A \subset X := T(\mathbf{R}^2)$ we can consider an *interior, exterior* and *boundary*(briefly, Bd) of A in X, where $T(\mathbf{R}^2)$ is the real plane with an *SAC* complex structure in terms of an edge to edge tiling and the *SST* structure T is formulated by the set

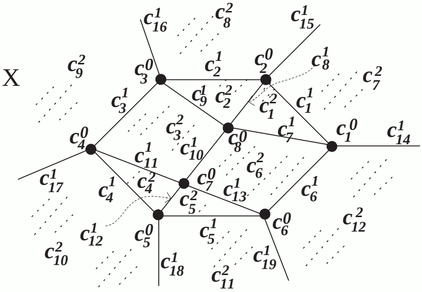
$$U := \{SN(x) \,|\, x \text{ is the element of } T(\mathbf{R}^2) \text{ in Figure 3}\}$$

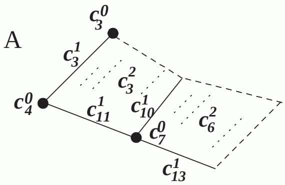
as a base.

Example

Consider an SST proposed on the SAC complex $X := T(\mathbf{R}^2)$ in Figure 3 and a subset A in Figure 3. Then, in (X, T) we observe that $Int(A) = \{c_3^2, c_6^2, c_{10}^1\}$ because $SN(c_j^2) = \{c_j^2\}, j \in \{3, 6\}$ and $SN(c_{10}^1) = \{c_3^2, c_{10}^1, c_6^2\}$; $Ext(A) = \{c_2^0, c_5^0, c_i^2, c_j^1 \mid i \in [1, 12]_{\mathbf{Z}} \setminus \{3, 6\}, j \in \{1, 2, 4, 5, 8, 12, [14, 19]_{\mathbf{Z}}\}$ and $Bd(A) = \{c_i^0, c_j^1 \mid i \in \{1, 3, 4, 6, 7, 8\}, j \in \{3, 6, 7, 9, 11, 13\}\}$.

Explanation of the notions of interior, exterior and boundary of an *SST*. (Fig.3)





Motivated by Example 3.1, under an *SST* we define a simple closed *SST*-curve, as follows:

Definition

We say that a simple closed *SST*-curve is an *SST* endowed with an *SAC* complex consisting of finite 0- and 1-cells such that each 0-cell has the only two adjacent 1-cells. Further, each 0-cell c^0 has $SN(c^0) = \{c^0, c_t^1 \mid c_t^1 \text{ is adjacent to } c^0\}$ and each 1-cell c^1 has $SN(c^1) = \{c^1\}$ in the *SST*.

Let us consider the real plane with an edge to edge tiling. For instance, consider the *SAC* complex $X := T(\mathbf{R}^2)$ in Figure 3 with the edge to edge tiling so that $(T(\mathbf{R}^2), N, dim) := T(\mathbf{R}^2)$ is an *SAC* complex. Then we can establish a Jordan curve theorem of an *SST*, as follows:

Theorem

Let $T(\mathbf{R}^2)$ be the real plane with an edge to edge tiling. Let $(T(\mathbf{R}^2), T)$ be an SST on $T(\mathbf{R}^2)$. Let C be a simple closed SST-curve on $T(\mathbf{R}^2)$. Then its complement, $T(\mathbf{R}^2) \setminus C$, consists of exactly two connected components. One of them is bounded (the interior) and the other is unbounded (the exterior) and the simple closed SST-curve is the boundary of each component.

Remark

(1) The Jordan curve theorem for SST has its own property because the type of open sets of an SST is different from those of the classical one. However, the function is the same with the classical one.
(2) In the above theorem both $Int(C)$ and $Ext(C)$ are open in the SST $(T(\mathbf{R}^2), T)$.

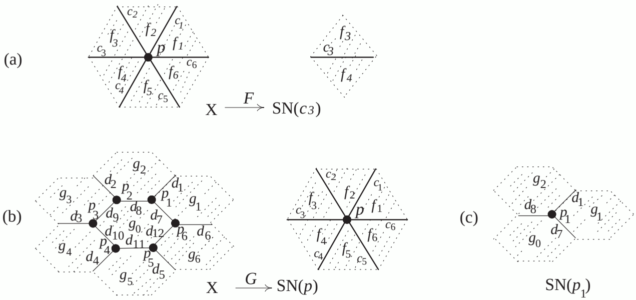
Continuity for a map between $SSTs$

By using the smallest neighborhood, we now define continuity of maps between $SSTs$

Definition

Let (X, T_1) and (Y, T_2) be $SSTs$ and let $f : (X, T_1) \to (Y, T_2)$ be a map. Then we say f is continuous at an element $x \in X$ if $f(SN(x, X)) \subset SN(f(x), Y)$.

Continuity of maps between *SSTs*. (Fig.4)



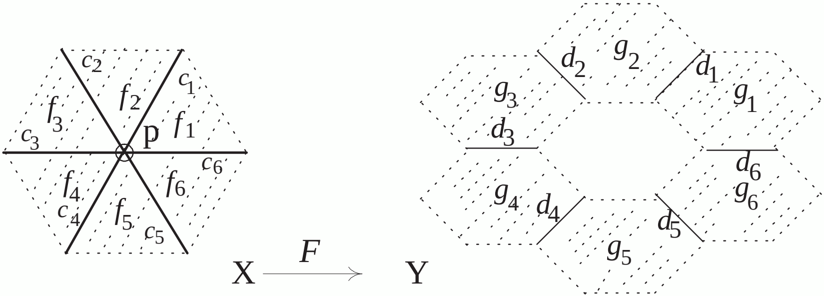
Homeomorphism of SST

Since continuity for maps f between $SSTs$ is related to the preservation of the neighborhood relation of the domain of f, it is meaningful to characterize homeomorphism between $SSTs$ in such a way:

Definition

Let X and Y be two $SSTs$. We say that a map $F : X \to Y$ is a homeomorphism if F is a continuous bijection and the inverse of F is also continuous.

Homeomorophism of maps between *SSTs*.(Fig.5)



Further works

- Dimension theory of new type of locally finite topological spaces
- Topological properties of an $SST \rightarrow$ semi-$T_{1/2}$-axiom
- Some properties of a homeomorphism in $CSST$
- Applications of the Jordan curve theorem for an SST
- Applications of SST in Discrete geometry and digital geometry
- Prove that Khalimsky nD space is equivalent to an SST.