Immersion Dimension and Topological complexity of projective product spaces

Jesús González

Mark Grant
joint with: Enrique Torres-Giese
Miguel Xicotencatl

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Ingredient 1

Farber's (2002) $TC(X) = secat\left(X^{[0,1]} \underset{ev}{\overset{s}{\rightleftarrows}} X \times X\right)$

Robotics (Motion Planning) real-time / in advance

$$X = \text{states of a robot}$$
$$X \times X = \text{initial-final states}$$
$$X^{[0,1]} = \text{robot transformations}$$

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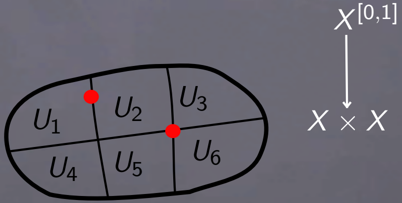
TC [normalized]

- homotopy invariant
- continuity instabilities in local motion planners



- $TC(X)$ detects mfld properties of X

Ingredient 2

Theorem (Farber-Tabachnikov-Yuzvinsky, 2002).

$$TC(\mathbb{R}P^n) = \begin{cases} Imm(\mathbb{R}P^n) & n \neq 1,3,7 \\ n & n = 1,3,7 \end{cases}$$

Theorem (G-Landweber, 2009).

$$TC^s(\mathbb{R}P^n) = Emb(\mathbb{R}P^n) \text{ for } n \neq 6,7,11,12,14,15$$

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<u>MORAL</u>: $\mathbb{R}P^n$ is a standard benchmark in AT.

Up to what extent is the above property part of a general phenomenon for mflds?

Eg:

- $TC(S^n) = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$
- $TC^s(S^n) = 2$

Spheres are highly connective!

Three examples closely related to $\mathbb{R}P^n$...

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Three examples closely related to $\mathbb{R}P^n$...

Ex. 1: 2-torsion odd-dim'l lens space $L^{2n+1}(2^e)$:

Theorem (Astey-Davis-G, 2003).

- $i :=$ smallest : $\exists \; S^{2n+1} \times S^{2n+1} \to S^i \quad \begin{cases} \mathbb{Z}/2\text{-biequiv} \\ \mathbb{Z}/2^e\text{-balanced} \end{cases}$

Then $Imm(L^{2n+1}(2^e)) = i$

Theorem (G, 2005).

- $t :=$ smallest : $\exists \; S^{2n+1} \times S^{2n+1} \to S^{2t+1} \quad \mathbb{Z}/2^e$-bieq.

Then $TC(L^{2n+1}(2^e)) = 2t + \begin{cases} 0 \\ 1 \end{cases}$

Corollary. $TC(L^{2n+1}(2^e)) + 1 \geq Imm(L^{2n+1}(2^e))$

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Ex. 2: Real Flag Mflds $\mathbb{R}F(1, ..., 1, m)$: $\mathbb{R}P^m = \mathbb{R}F(1, m)$

- motion planning of box frames
- π_1 is an elementary abelian 2-group.

Theorem (G-Guitierrez-Torres, 2012).

- $TC(\mathbb{R}F(1,1,m)) \geq \begin{cases} 4m+1, & m = 2^e - 1 \\ 4 \cdot 2^e - 2, & 2^e \leq m \leq 2^{e+1} - 1 \end{cases}$

- $TC(\mathbb{R}F(1,1,1,m)) \geq \begin{cases} 5m+6, & m = 2^e - 2 \\ 6m+2, & m = 2^e - 1 \\ 6 \cdot 2^e - 3, & 2^e \leq m \leq 2^{e+1} - 3, e \geq 2 \end{cases}$

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Eg:

$$\begin{aligned}
TC(\mathbb{R}F(1,1,1)) &\geq 5 > 4 = Imm(\mathbb{R}F(1,1,1)) \\
TC(\mathbb{R}F(1,1,1,1)) &\geq 8 > 7 = Imm(\mathbb{R}F(1,1,1,1)) \\
TC(\mathbb{R}F(1,1,2)) &\geq 6 = Imm(\mathbb{R}F(1,1,2)) \\
TC(\mathbb{R}F(1,1,3)) &\geq 13 > 10 = Imm(\mathbb{R}F(1,1,3)) \\
TC(\mathbb{R}F(1,1,1,2)) &\geq 16 > 10 = Imm(\mathbb{R}F(1,1,1,2)) \\
TC(\mathbb{R}F(1,1,1,3)) &\geq 20 > 15 = Imm(\mathbb{R}F(1,1,1,3)) \\
TC(\mathbb{R}F(1,1,1,4)) &\geq \underbrace{21 = Imm(\mathbb{R}F(1,1,1,4))}_{\text{Lam + Stong}}
\end{aligned}$$

... so it looks like

$$TC(M) + 1 \geq Imm(M)$$

for M a close relative of $\mathbb{R}P^n$...

Ingredient 3 $\left(\substack{=\text{Ex. } 3\\ =\text{main result}}\right)$:

$P_{(n_1,\dots,n_r)} = S^{n_1} \times \cdots \times S^{n_r}/\mathbb{Z}/2$-diagonally
$\pi_1 = \mathbb{Z}/2$ generically.

Theorem (G-Grant-Torres-Xicotencatl, 2012).
Let $\overline{n} = (n_1, \dots, n_r)$ with $n_1 \leq n_i \forall i$. Then

$$TC(P_{\overline{n}}) < (TC(P^{n_1}) + 1)(r + e)$$

where $e = \#$ of even dim'l spheres S^{n_i} con $n_i \geq 2$

- $TC(P_{\overline{n}})$ can be much lower than $dim(P_{\overline{n}}) < Imm(P_{\overline{n}})$
- **Conjecture.** $TC(P_{\overline{n}}) \approx F(TC(P^{n_1}), r, e)$

Eg: $TC(P_{\overline{n}}) = TC(P^{n_1}) + r - 1$ provided $\begin{cases} n_1 = 2^l \\ \nu(n_i + 1) \geq \phi(2^l) \end{cases}$

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Methods

$$S^{n_2} \times \cdots \times S^{n_r} \longrightarrow P_{\overline{n}} = S^{n_1} \underset{\mathbb{Z}/2}{\times} (S^{n_2} \times \cdots \times S^{n_r}) \longrightarrow P^{n_1}$$

- cat is submultiplicative wrt fibrations:

$$cat(P_{\overline{n}}) < (cat(P^{n_1}) + 1)(cat(S^{n_2} \times \cdots \times S^{n_r}) + 1)$$

- unknown for TC; true for the fibration above.

Theorem (Colman-Grant, 2012, arXiv 1205.0166).
For a G-principal fibration $P : E \to B$ and a G-space F :

$$TC(E \times_G F) < (TC(B) + 1)(TC_G(F) + 1)$$

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Definition. For a G-space F

$$TC_G(F) = secat_G(F^{[0,1]} \longrightarrow F \times F)$$

where G acts via: composition $\rightarrow$ diagonally.

Thus, we need: $TC(S^{n_2} \times \cdots \times S^{n_r}) = TC_{\mathbb{Z}/2}(S^{n_2} \times \cdots \times S^{n_r})$,

But:

- $TC(X) \leq TC_G(X)$
- $TC_G(X \times Y) \leq TC_G(X) + TC_G(Y)$
- $TC_{\text{antipodal}}(S^n) = TC(S^n) = \begin{cases} 1, & n \text{ odd} \\ 2, & n \text{ even} \end{cases}$

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Moral: Equalities $TC = Imm$ and $TC^s = Emb$ for $\mathbb{R}P^m$ are really more on the ~~accidental~~ exceptional side!