# Topological degree calculation based on interval arithmetic

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This talk describes

- an algorithm for degree computation that was implemented and is available
- application to satisfiability of systems of equations.

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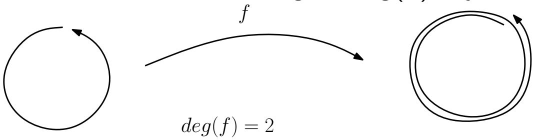

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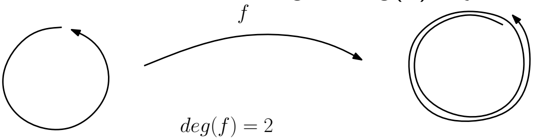


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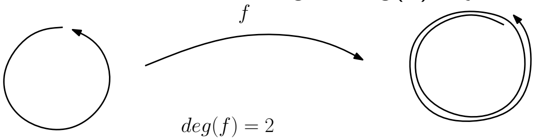


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- For a continuous map $f : M \to N$ between compact oriented manifolds of the same dimension $n$, the degree $\deg(f)$ is defined (e.g.) by

$$\deg(f) \int_N \omega = \int_M f^*\omega$$

for any $n$-form $\omega$.

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Due to the last property, we will be interested only in calculating $\deg(f, \Omega, 0)$.

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- Murashige, 2006: implementation of an algorithm using computational homology – computational results very slow
- Homology computation packages usually require simplicial/cubical setting
- Kearfott, 2004: fast algorithm working in high dimensions, but only for complex functions $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^n$

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- expression $xy + 1$,
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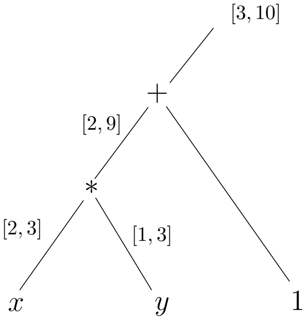

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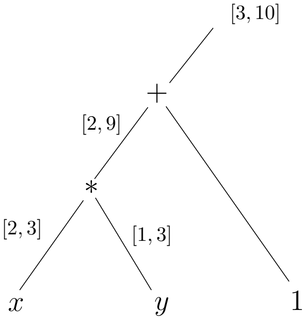


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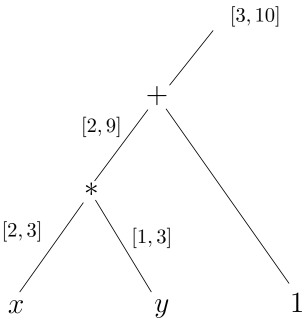


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- If lower bound of interval greater than zero then $f > 0$ on $B = [2, 3] \times [1, 3]$.
- if upper bound of interval less than zero then $f < 0$ on $B$.

# Interval arithmetic

### Definition

An interval computable function is a function $f : B \to R$ and an algorithm $[f]$ that computes, for a box $C \subseteq B$, an interval $[f](B)$ s.t.

- $f(C) \subseteq [f](C)$
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Expressions constructed from $+ \; - \; * \, / \; \mathit{sin} \; \mathit{cos} \; \mathit{exp} \ldots$ are interval-computable functions.

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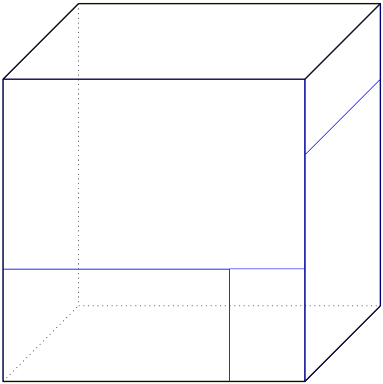

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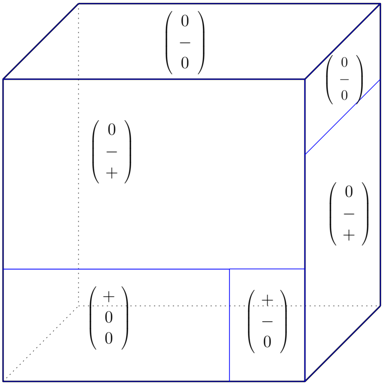

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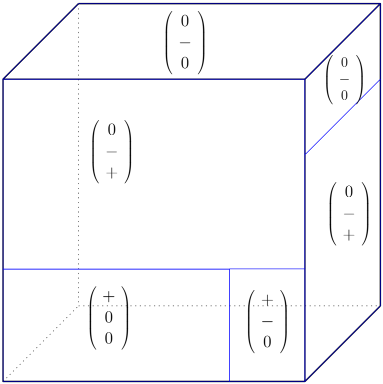

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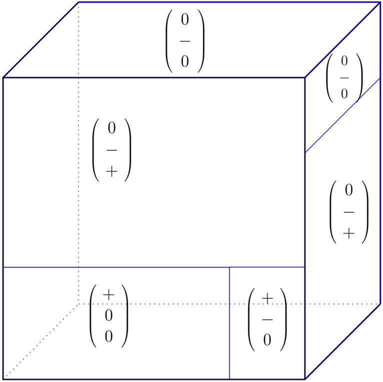

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*Sign information determines the degree* $\deg(f, B, 0)$ *uniquely*
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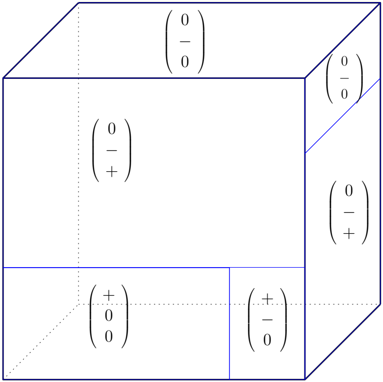

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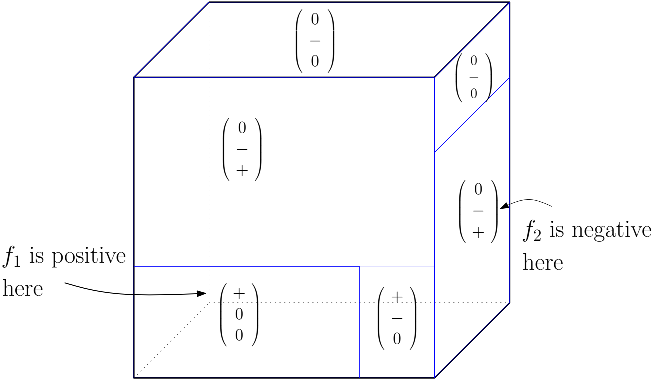

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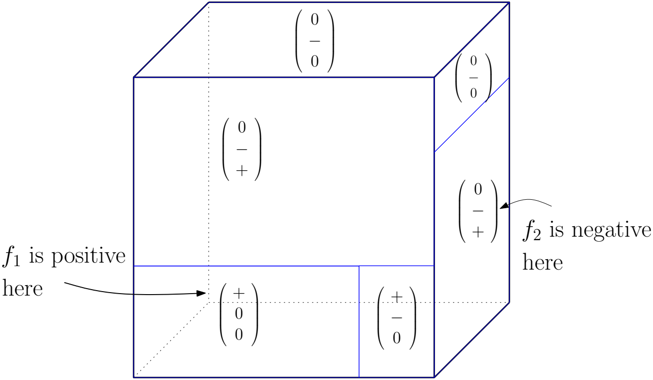

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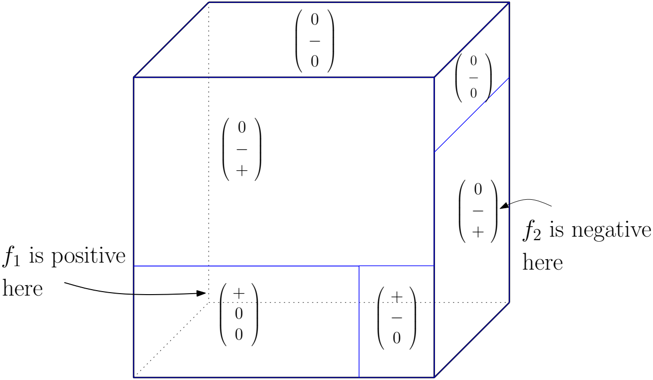
Basic tool: Interval arithmetic



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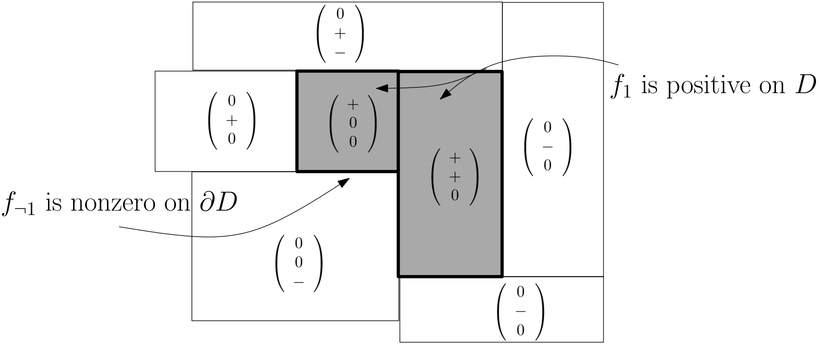

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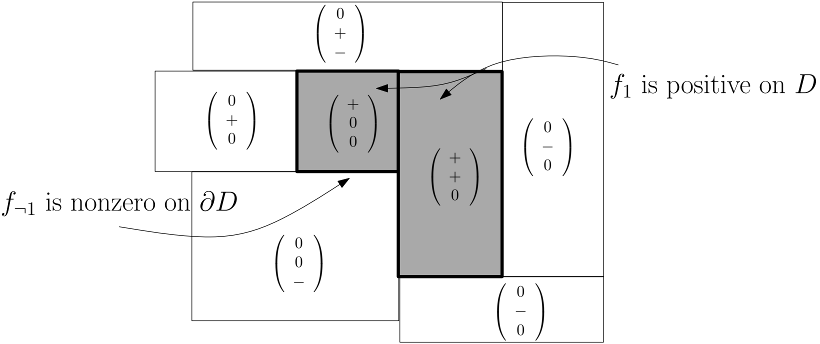

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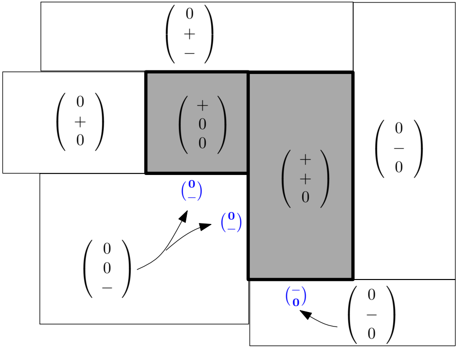

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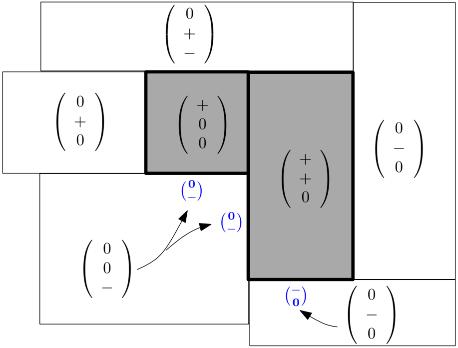
- Recursion: $\deg(f, B, 0) = s\,(-1)^{l+1} \sum_j \deg(f_{\neg l}, D^j, 0)$



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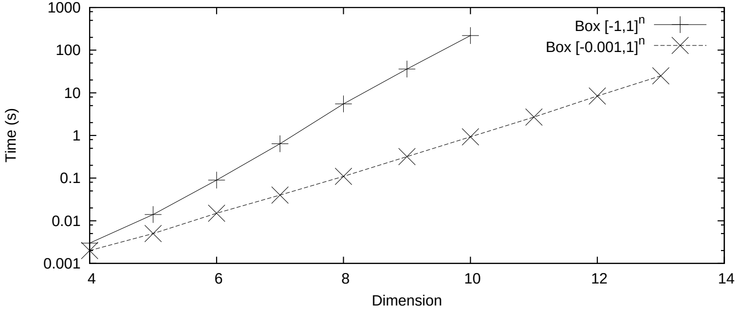
- Keeping track of orientation and induced orientation on boundary boxes
- Handling lower-dimensional intersection with neighboring boxes
- Choosing coordinate $l$ and sign $s$ in an optimal way (s.t. the number of boxes where $f_l$ has sign $s$ is minimal)

# Computational Experiments

$$
\begin{aligned}
f_1 &= x_1^2 - x_2^2 - \ldots - x_n^2 \\
f_2 &= 2x_1x_2 \\
&\ldots \\
f_n &= 2x_1x_n.
\end{aligned}
$$

with degree 2 for $n$ even and 0 for $n$ odd, if $0 \in B$. Times for calculating $\deg(f, B, 0)$ for $B = [-1, 1]^n$ and $B = [-0.001, 1]^n$.



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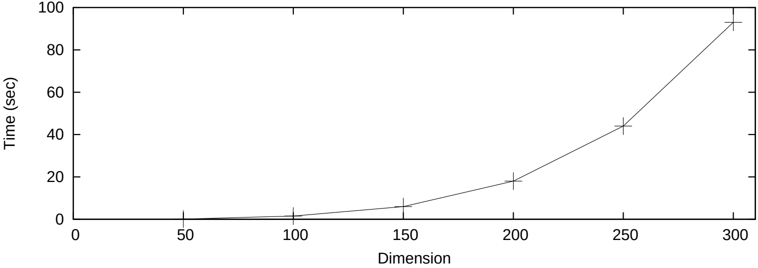
- Computation times for $\deg(\mathrm{id}, [-1,1]^n, 0)$ :



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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S^0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $S^1$ | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $S^2$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_{12}\times\mathbb{Z}_2$ |
| $S^3$ | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_{12}\times\mathbb{Z}_2$ |
| $S^4$ | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}\times\mathbb{Z}_{12}$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_{24}\times\mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^3$ |
| $S^5$ | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{30}$ | $\mathbb{Z}_2$ |
| $S^6$ | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{60}$ |
| $S^7$ | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | 0 | $\mathbb{Z}_2$ |

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Does there exists a solution in $B$?

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- If $f(x) = 0$, is robustly satisfiable, then there exists $U \subseteq B$ s.t. $\deg(f, U, 0) \neq 0$.

[Franek, Ratschan, Zgliczynski: Quasi-decidability of a Fragment of the Analytic First-order Theory of Real Numbers, 2012]

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More general: for $f : B \subseteq \mathbb{R}^m \to \mathbb{R}^n$, $0 \notin f(\partial B)$.
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  - The work of Krcal, Matousek, Sergeraert etc $\Longrightarrow$ possible undecidability for general $m, n$ (?).

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Thanks for you attention.