ANALYSIS, COMBINATORICS

COMPUTATION

EULER NUMBER \( X \)

\[ V - E + F = 2 \quad (\text{on sphere}) \]

\[ = 2 - 2g \quad (\text{on surface of genus } g) \]

HIGHER-DIMS

\[ \sum_{2=0}^{n} (-1)^2 c^2 = \sum_{2=0}^{n} (-1)^2 r^2 = X \]

\( c^2 = \) number of 2-cells

COMBINATORIAL

\( r^2 = \text{dim } H^2 \)

\( H^2 = (C^0)\text{-Homology group} \)

TOPOLOGICAL

IN VARIANTS

\[ \pi_1 \]
INTEGRAL FORMULA

RIEMANNIAN METRIC

dim 2: GAUSS (SCALAR) CURVATURE $\kappa$

$\kappa = \frac{1}{2\pi} \int \kappa$

(ALSO IN HIGHER-DIM)

DIFFERENTIAL FORMULA

$\Omega^2$ DIFFERENTIAL 2-FORMS

$d: \Omega^2 \rightarrow \Omega^{p+2}$ EXTERIOR DERIVATIVE

DE RHAM COMPLEX

$0 \rightarrow \Omega^0 \rightarrow \Omega^1 \rightarrow \Omega^2 \rightarrow \cdots \rightarrow \Omega^n \rightarrow 0$

$H^2 =$ COHOMOLOGY OF THIS COMPLEX

HODGE THEORY RIEMANNIAN METRIC

$d^* =$ ADJOINT OF $d$

HODGE LAPLACIAN $\Delta = d d^* + d^* d$

$\Delta \phi = 0$ HARMONIC 2-FORM $\mathcal{H}^2$

HODGE THEOREM $\mathcal{H}^2 \cong \mathcal{H}^2$

$\Rightarrow \kappa = \sum (-1)^p \dim \mathcal{H}^p$
LAPLACE TYPE OPERATORS

2^{ND} ORDER ELLIPTIC

in \( \mathbb{R}^n \)

\(- \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \)

HODGE \( \Delta \)

DIRAC TYPE OPERATORS

1^{ST} ORDER ELLIPTIC (SYSTEMS)

in \( \mathbb{R}^n \)

\(- \sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i} \)

\( A_i = -1 \)

\( A_i A_j = -A_j A_i \quad (c+i) \)

SQUARE-ROOT OF

LAPLACE-TYPE

dim 1

\[ \frac{\partial}{\partial x} \]

dim 2

CAUCHY-RIEMANN

\[ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \]

ON DIFFERENTIAL FORMS

\[ \Omega = \Theta \Omega^2 \]

SPINORS

\[ (d+d^*) \]

\[ (d+d^*)^2 = dd^* + d^*d \quad \text{HODGE} \Delta \]

\[ (d^*d)^2 = 0 \]
INDEX THEORY

\[ d^{+}d^{'} = \Omega^{ev} \rightarrow \Omega^{odd} \]

INDEX = dim \( \Omega^{ev} \) - dim \( \Omega^{odd} \) = \( \chi \)

SIMPLEST EXAMPLE OF AN INDEX THEOREM (GIVEN BY INTEGRAL FORMULA)

BUT ATYPICAL: VERY ROBUST

COMBINATORIAL VERSION PRECISE

AT ALL LEVELS

IN GENERAL

1) FOR DIRAC-TYPE OPERATOR \( D \)

\( \mathcal{N}^{+}, \mathcal{N}^{-} \) (NULL-SPACES OF \( D^{+}D \))

HAVE DIMS WHICH CAN "JUMP"

THOUGH DIFFERENT (INDEX) IS TOPOLOGICAL INVARIANT.

2) COMBINATORIAL APPROXIMATIONS TO \( D \) "NOT PRECISE"
Examples of (4)

a) \[ \text{dim} \ 2 \quad \text{SURFACE} \quad \text{CONFORMAL (COMPLEX)} \]
\[ d = 2 + 3 = \left( \frac{3}{2}, \frac{3}{2} \right) \]

COMPLEX DE RHAM
\[ \Omega^3 \rightarrow \Omega^0 \]
\[ H^0 = \text{CONSTANTS} \quad H^1 = \text{Holomorphic DIFFS} \]
\[ \text{index} = 1 - q \quad \left( \geq \frac{1}{2}q \right) \]

But what is combinatorial version?

b) \( \text{dim} \ 4 \quad \text{(ORIENTED)} \)
\[ H^2 \quad \text{HAS QUADRATIC FORM} \]
(\( H^2 \) "INTERSECTION FORM")

\[ \text{METRIC} \]
\[ H^2 = \mathfrak{h}^2 = H^2_+ + \Theta H_- \]
\[ \times \phi = -\phi \quad \times \phi = -\phi \]

\[ \text{Signature} \quad \sigma = \text{dim} H^2_+ - \text{dim} H^2_- \]

\[ \text{TOPOLOGICAL INVARIANT} \]
\[ \text{INTEGRAL FORMULA} = \frac{1}{3} \sqrt{p_3} \]

(PONTJAGIN FORM)
$I$ can be viewed as index of operator $\alpha + \alpha^*: \Omega^x \to \Omega^y$ where $\Omega = \Omega^x \cap \Omega^y$ is decomposition given by $I$

Note $+$ maps $\Omega^0 \leftrightarrow \Omega^y$

$\Omega^1 \leftrightarrow \Omega^3$

And decomposes $\Omega^2$ into $\text{dual}$ $\text{dual}$

$\Rightarrow$ Index $=$ Signature

But finding combinatorial version runs into problems

Cellular $\to$ Dual Cells

Refinement

Infinite regression!
COMBINATORIAL INDEX PROBLEM

\[ D : E \rightarrow F \quad \text{ELLiptic (Dirac-\text{TYPE}) DIFF. OPERATOR} \]

WITH its ADJOINT \( D^* \)

HAD A PRECISE COMBINATORIAL VERSION AT ALL LEVELS

\[ D_n : E_n \rightarrow F_n \]

THEN

\[ \text{index } D_n = \dim E_n - \dim F_n \quad \text{(because we are in finite dim)} \]

AS \( n \to \infty \) WE WANT TO RECOVER DIFF OPERATOR STORY

SO \( \text{INDEX } D_n \rightarrow \text{INDEX } D \)

IT SEEMS WE NEED TO FEED THE ANSWER INTO THE COMB. APPROX
BREAKING THE SYMMETRY

Given \( D : E \rightarrow f \) elliptic

Perhaps we can find finite approximations \( D_n \rightarrow D_n' \) to

\( D \rightarrow D^* \) which are not adjoints

with \( D_n \rightarrow D \rightarrow D_n' \rightarrow D^* \)

Then we need to study how

\( (D_n^* D_n) \rightarrow D^* D \)

on how its eigenvalues behave

as \( n \rightarrow \infty \).

KEY OBSERVATION "physics"

Focus not on \( 0 \)-eigenvalue of \( D_n \)

but on low-lying eigenvalues

those which \( \rightarrow 0 \) as \( n \rightarrow \infty \)

J. PACHOS (LEEDS)
Computational Question

How do we identify the "low-lying" eigenvalues in some precise way?

Note for Dirac-type operators arising in Riemannian geometry:
The key data (metric + connection) \[\Rightarrow\] curvature (of base + bundle)

Perhaps we need an approximation so that all curvatures over cells are "small"?
(1) **EXAMPLE OF "JUMPS"**

\[ \dim 2 \text{ Complex Riemann Surface } X \]

**Holomorphic Line-Bundle** \( L \)

\[ \dim H^0(X,L) \text{ can vary as complex moduli of } L \text{ vary } (g \geq 1) \]

But \[ \dim H^0 - \dim H^2 \text{ topological } = 1 + \deg(L) \]

**Not** here variation is bounded

In **higher-dims** is unbounded

**Question** how will this affect

**Combinatorial approximations**?
EIGENVALUES & CURVATURE

FOR LAPLACE-BELTRAMI OPERATOR
(= HODGE ON SCALAR FUNCTIONS)

ESTIMATES ON FIRST NON-ZERO
EIGENVALUE $\lambda_1$

CHEEGER ISOFLPERIMETRIC CONSTANT
M. COMPACT RIEMANNIAN

\[ \frac{\int E}{\min \left\{ V(A), V(B) \right\} } \]

\[ E \subset M, \dim E = n-1 \]

\[ V = n\text{-Volume} \]

\[ S = (n-1)\text{-volume "area"} \]

CHEEGER INEQUALITY

\[ \lambda_1(M) \geq \frac{\lambda^2(M)}{4} \]

"SHARP"
If Riemannian curvature bounded below \( - (n-1) \alpha^2 \) \((\alpha > 0)\), then

\[ \lambda_1(M) \leq 2 \alpha (n-1) \kappa(M) + \text{log}^2(M) \]

Buser inequality

Story more complicated for other geometric operators

Probably large literature ??
Focus on key example of signature of 4-manifold

[Note results of Belfand-Macpherson]

Steps

1) Set up comb. approximations $\Delta_N, \Delta'_N$

2) Define "low-lying" eigenvalues of these

3) Use curvature estimates to show these eigenvalues are well-defined for large $N$

6) Conclude

signature = $R_N - R'_N$

$R, R'$ number of low-lying eigenvalues of $\Delta_N, \Delta'_N$

Hope to get useful formula!!